

Gravitational and axial anomalies for generalized Euclidean Taub-NUT metrics

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Abstract

The gravitational anomalies are investigated for generalized Euclidean Taub-NUT metrics which admit hidden symmetries analogous to the Runge-Lenz vector of the Kepler-type problem. In order to evaluate the axial anomalies, the index of the Dirac operator for these metrics with the APS boundary condition is computed. The role of the Killing-Yano tensors is discussed for these two types of quantum anomalies.

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1 Introduction

In the case of gravitational interaction, a consistent perturbative quantization is not available, even if there exist no fermions. It is of crucial importance in the construction of any quantum theory for gravitation to understand the problem of anomalies which can affect the conservation laws.

In the present paper we shall investigate the quantum anomalies with regard to quadratic constants of motion in some explicit examples - the Euclidean Taub-Newman-Unti-Tamburino (Taub-NUT) space and its generalizations as it was done by Iwai and Katayama [1, 2, 3, 4].

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Hidden symmetries are encapsulated into Stäckel-Killing (S-K) tensors. Here we consider symmetric tensors $k_{\mu\nu} = k_{\nu\mu}$ satisfying the S-K equation

$$k_{(\mu\nu;\lambda)} = 0 \quad (1)$$

where a semicolon precedes an index of covariant differentiation. For any geodesic with tangent (momentum vector) p_μ a S-K tensor generates a quadratic constant along geodesic,

$$K = k^{\mu\nu} p_\mu p_\nu, \quad p_\mu = g_{\mu\nu}(x) \dot{x}^\nu, \quad (2)$$

where g is the metric tensor and the over-dot denotes the ordinary proper time derivative. If we are only interested in the geodesic motion of classical scalar particles, then eq.(1) is the necessary and sufficient condition for the existence of a quadratic constant of motion (2), as can be seen from the Poisson bracket of K with the Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \quad (3)$$

Passing from the classical motion to the hidden symmetries of a quantized system, the corresponding quantum operator analog of the quadratic function (2) is [5, 6]:

$$\mathcal{K} = D_\mu k^{\mu\nu} D_\nu \quad (4)$$

where D_μ is the covariant differential operator on the curved manifold. Working out the commutator of (4) with the scalar Laplacian

$$\mathcal{H} = D_\mu D^\mu = D_\mu g^{\mu\nu} D_\nu \quad (5)$$

we get

$$\begin{aligned} [D_\mu D^\mu, \mathcal{K}] = & 2k^{\mu\nu;\lambda} D_{(\mu} D_\nu D_{\lambda)} + 3k^{(\mu\nu;\lambda)}_{;\lambda} D_{(\mu} D_{\nu)} \\ & + \left\{ \frac{1}{2} g_{\lambda\sigma} (k_{(\lambda\sigma;\mu);\nu} - k_{(\lambda\sigma;\nu);\mu}) - \frac{4}{3} k_\lambda^{[\mu} R^{\nu]\lambda} \right\}_{;\nu} D_\mu \end{aligned}$$

where $R_{\mu\nu}$ is the Ricci tensor.

In the classical case, the fact that $k_{\mu\nu}$ is a S-K tensor satisfying eq.(1), assures the conservation of (2). Concerning the hidden symmetry of the quantized system, the above commutator does not vanish on the strength of (1). Taking into account eq.(1) we get:

$$[\mathcal{H}, \mathcal{K}] = -\frac{4}{3} \{k_\lambda^{[\mu} R^{\nu]\lambda}\}_{;\nu} D_\mu \quad (6)$$

which means that in general the quantum operator \mathcal{K} does not define a genuine quantum mechanical symmetry [7]. On a generic curved spacetime there appears a *gravitational quantum anomaly* proportional to a contraction of the S-K tensor $k_{\mu\nu}$ with the Ricci tensor $R_{\mu\nu}$.

In other respects, the behavior of the covariant derivatives of the spinor theory is also dependent on the Ricci tensor. The standard Dirac theory is formulated in local frames where the tetrad fields $e(x)$ and $\hat{e}(x)$ determine the form of the point-dependent Dirac matrices γ^μ (obeying $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} \mathbf{1}$) and the spin connection Γ_μ^{spin} of the spinor covariant derivatives which act on the spinor field ψ as $D_\mu \psi = \partial_\mu \psi + \Gamma_\mu^{spin} \psi$ [8]. Moreover, the covariant derivatives commute with γ^μ and satisfy

$$[D_\mu, D_\nu] \psi = \frac{1}{4} R_{\alpha\beta\mu\nu} \gamma^\alpha \gamma^\beta \psi,$$

where $R_{\alpha\beta\mu\nu}$ is the Riemann curvature tensor. Hereby one deduces the properties of the *standard* Dirac operator $\mathcal{D} = \gamma^\mu D_\mu$. Using the identity $R_{\alpha\beta\mu\nu} \gamma^\beta \gamma^\mu \gamma^\nu = -2R_{\alpha\nu} \gamma^\nu$ one finds the commutation rules

$$[D_\mu, \mathcal{D}] \psi = -\frac{1}{2} R_{\mu\nu} \gamma^\nu \psi$$

which show that the squared Dirac operator,

$$\mathcal{D}^2 \psi = \left(\mathcal{H} - \frac{1}{4} R \mathbf{1} \right) \psi,$$

coincides with \mathcal{H} only when the Ricci scalar R vanishes.

Hence the conclusion is that in the Ricci-flat manifolds with $R_{\mu\nu} = 0$ three phenomena occur simultaneously: (I) the scalar quantum anomaly disappears, (II) the Dirac operator becomes the exact square root of the Laplace operator and (III) the covariant derivatives commute with \mathcal{D} . In this case the operators D_μ are *conserved* and could be taken as momentum operators even though they do not commute among themselves.

In general, when the manifold is not Ricci-flat the operators constructed from symmetric S-K tensors are a source of gravitational anomalies for scalar fields. However, when the S-K tensors admit a decomposition in terms of antisymmetric tensors Killing-Yano (K-Y) [9] the gravitational anomaly is absent.

The K-Y tensors are profoundly connected with supersymmetric classical and quantum mechanics on curved spaces where such tensors do exist [10]. The K-Y tensors play an important role in theories with spin and especially in the Dirac theory on curved spacetimes where they produce first-order differential operators, called Dirac-type operators, which anticommute with the standard Dirac one, D [9]. When the K-Y tensors enter as square roots in the structure of several second-rank S-K tensors, they generate conserved quantities in pseudo-classical models for fermions [10] or conserved operators in Dirac theory which commute with D .

In the pseudo-classical approach [10] of the fermions, the absence of the K-Y tensors hampers the evaluation of the spin contribution to the conserved quantities. Passing to Dirac equation in a curved background, the lack of the K-Y tensors makes impossible the construction of Dirac-type operators and hidden quantum conserved operators commuting with the standard Dirac one.

Having in mind that the K-Y tensors prevent the appearance of gravitational anomalies for scalar field and on the other hand their connection with supersymmetries and Dirac-type operators, it is natural to investigate their role in axial anomalies.

The importance of anomalous Ward identities in particle physics is widely appreciated. The anomalous divergence of the axial vector current in a background gravitational field was large discussed in the literature and directly related with the index theorem. In even-dimensional spaces one can define the index of a Dirac operator as the difference in the number of linearly independent zero modes with eigenvalue $+1$ and -1 under γ_5 . The index is useful as a tool to investigate topological properties of the space, as well as in computing anomalies in quantum field theory.

In this paper we want to investigate the continuous transition from the case in which a hidden symmetry is described by a S-K tensor which can be written as a symmetrized product of K-Y tensors to the situation in which the K-Y tensors are absent.

In Section 2 we verify explicitly that for extended Taub-NUT metric the commutator (6) does not vanish and consequently there are gravitational anomalies.

In Section 3 we consider the Dirac operator on extended Taub-NUT spaces. In next Sections we compute the index of the Dirac operator for the generalized Taub-NUT metrics with the APS boundary condition and we find these metrics do not contribute to the axial anomaly at least for not too large deformations of the standard Taub-NUT metric. This result stand in contrast with the quantum anomalies for scalar fields discussed in Section 2. The result is natural since the index of an operator is unchanged under continuous deformations of that operator. In our case this would amount to a continuous change in the metric and the boundary condition. However for larger deformations of the metric there could appear discontinuities in the boundary condition and therefore the index could present jumps. Our formula for the index involves a computable number-theoretic quantity depending on the coefficients of the metric.

In Section 6 we point out some open problems in connection with unbounded domains. The last Section contains some concluding remarks.

2 Gravitational anomalies in extended Taub-NUT spaces

The Euclidean Taub-NUT metric is involved in many modern studies in physics [11, 12]. From the viewpoint of dynamical systems, the geodesic motion in Taub-NUT metric is known to admit a Kepler-type symmetry [13, 14, 15, 16]. One can actually find the so called Runge-Lenz vector as a conserved vector in addition to the angular momentum vector. As a consequence, all the bounded trajectories are closed and the Poisson brackets among the conserved vectors give rise to the same Lie algebra as the Kepler problem, depending on the energy. Thus the Taub-NUT metric provides a non-trivial generalization of the Kepler problem.

Iwai and Katayama [1, 2, 3, 4] generalized the Taub-NUT metric so that it still admit a Kepler-type symmetry.

2.1 Extended Taub-NUT spaces

The Euclidean Taub-NUT space is a special member of the family of four-dimensional manifolds equipped with the isometry group $G_{iso} = SO(3) \otimes U(1)$. These geometries can be easily constructed defining the line elements in local charts with spherical coordinates $(r, \theta, \varphi, \chi)$; among them the first three are the usual spherical coordinates of the vector $\vec{x} = (x^1, x^2, x^3)$, with $|\vec{x}| = r$, while χ is the Kaluza-Klein extra-coordinate of this chart. The spherical coordinates can be associated with the Cartesian ones (x^1, x^2, x^3, x^4) where $x^4 = -\mu(\chi + \varphi)$ is defined using an arbitrary constant $\mu > 0$.

The group $SO(3) \subset G_{iso}$ has three independent one-parameter subgroups, $SO_i(2)$, $i = 1, 2, 3$, each one including rotations $\mathfrak{R}_i(\phi)$, of angles $\phi \in [0, 2\pi)$ around the axis i . With this notation any rotation $\mathfrak{R} \in SO(3)$ in the usual Euler parametrization reads $\mathfrak{R}(\alpha, \beta, \gamma) = \mathfrak{R}_3(\alpha)\mathfrak{R}_2(\beta)\mathfrak{R}_3(\gamma)$. Moreover, we can write $\vec{x} = \mathfrak{R}(\varphi, \theta, 0)\vec{x}_o$ where the vector $\vec{x}_o = (0, 0, r)$ is invariant under $SO_3(2)$ rotations which form its little group ($\mathfrak{R}_3\vec{x}_o = \vec{x}_o$). The main point is to define the action of two arbitrary rotations, $\mathfrak{R} \in SO(3)$ and $\mathfrak{R}_3 \in SO_3(2) \sim U(1)$, in the spherical charts, $(\mathfrak{R}, \mathfrak{R}_3) : (r, \theta, \varphi, \chi) \rightarrow (r, \theta', \varphi', \chi')$, such that

$$\mathfrak{R}(\varphi', \theta', \chi') = \mathfrak{R}(\varphi, \theta, \chi)\mathfrak{R}_3^{-1}. \quad (7)$$

Hereby it results that the Cartesian coordinates transform under rotations $\mathfrak{R} \in SO(3)$ as

$$\begin{aligned} \vec{x} &\rightarrow \vec{x}' = \mathfrak{R}\vec{x}, \\ x^4 &\rightarrow x'^4 = x^4 + h(\mathfrak{R}, \vec{x}), \end{aligned}$$

where the function h is given in Ref. [17]. Thus, the vector \vec{x} transforms according to an usual linear representation but the transformation of the fourth Cartesian coordinate is governed by a representation of $SO(3)$ induced by $SO_3(2)$ [17]. Furthermore, we observe that the 1-forms

$$d\Omega(\varphi, \theta, \chi) = \mathfrak{R}(\varphi, \theta, \chi)^{-1}d\mathfrak{R}(\varphi, \theta, \chi) \in so(2)$$

transform independently on \mathfrak{R} as

$$(\mathfrak{R}, \mathfrak{R}_3) : d\Omega(\varphi, \theta, \chi) \rightarrow d\Omega(\varphi', \theta', \chi') = \mathfrak{R}_3 d\Omega(\varphi, \theta, \chi) \mathfrak{R}_3^{-1}$$

finding that, beside the trivial quantity $ds_1^2 = dr^2$, there are two types of line elements invariant under G_{iso} ,

$$\begin{aligned} ds_2^2 &= -\langle d\Omega(\varphi, \theta, \chi)^2 \rangle_{33} = d\theta^2 + \sin^2 \theta d\varphi^2, \\ ds_3^2 &= -\frac{1}{2} Tr [d\Omega(\varphi, \theta, \chi)^2] = d\theta^2 + \sin^2 \theta d\varphi^2 + (d\chi + \cos \theta d\varphi)^2. \end{aligned}$$

The conclusion is that the most general form of the line element invariant under G_{iso} is given by the linear combination $f_1(r)ds_1^2 + f_2(r)ds_2^2 + f_3(r)ds_3^2$ involving three arbitrary functions of r , f_1 , f_2 and f_3 .

Here it is worth pointing out that the above metrics are related to the Berger family of metrics on 3-spheres [18]. These are introduced starting with the Hopf fibration $\pi_H : S^3 \rightarrow S^2$ that defines the vertical subbundle $V \subset TS^3$ and its orthogonal complement $H \subset TS^3$ with respect to the standard metric g_{S^3} on S^3 . Denoting with g_H and g_V the restriction of g_{S^3} to the horizontal, respectively the vertical bundle, one finds that the corresponding line elements are $ds_H^2 = \frac{1}{4}ds_2^2$ and $ds_V^2 = \frac{1}{4}(ds_3^2 - ds_2^2)$. For each constant $\lambda > 0$ the Berger metric on S^3 is defined by the formula

$$g_\lambda = g_H + \lambda^2 g_V. \quad (8)$$

In what follows we restrict ourselves to the *extended* Taub-NUT manifolds whose metrics are defined on $\mathbb{R}^4 - \{0\}$ by the line element

$$\begin{aligned} ds_K^2 &= g_{\mu\nu}(x)dx^\mu dx^\nu \\ &= f(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2) + g(r)(d\chi + \cos \theta d\varphi)^2 \end{aligned} \quad (9)$$

where the angle variables (θ, φ, χ) parametrize the sphere S^3 with $0 \leq \theta < \pi, 0 \leq \varphi < 2\pi, 0 \leq \chi < 4\pi$, while the functions

$$f(r) = \frac{a + br}{r}, \quad g(r) = \frac{ar + br^2}{1 + cr + dr^2}. \quad (11)$$

depend on the arbitrary real constants a, b, c and d . This line element can be written in terms of the Berger metrics as

$$ds_K^2 = (ar + br^2) \left(\frac{dr^2}{r^2} + 4ds_{\lambda(r)}^2 \right) \quad (12)$$

where $ds_{\lambda(r)}^2 = (g_{\lambda(r)})_{\mu\nu} dx^\mu dx^\nu$ and

$$\lambda(r) = \frac{1}{\sqrt{1 + cr + dr^2}}. \quad (13)$$

If one takes the constants

$$c = \frac{2b}{a}, \quad d = \frac{b^2}{a^2} \quad (14)$$

with $4m = \frac{a}{b}$, the extended Taub-NUT metric becomes the original Euclidean Taub-NUT metric up to a constant factor. In the original Kaluza-Klein context the Taub-NUT parameter m is positive.

By construction, the spaces with the metric (9) have the isometry group G_{iso} and, therefore, they must have four Killing vectors k_A^μ labeled by an index $A = 1, 2, 3, 4$ depending on the parametrization of G_{iso} . The usual constants of motion for particles moving in these backgrounds are linear in the four momentum p_μ ,

$$J_A = k_A^\mu p_\mu. \quad (15)$$

For a particle in extended Taub-NUT backgrounds the corresponding constants of motion [12, 13, 14, 15, 16] consist of a quantity which, for negative mass models, can be interpreted as the “relative electric charge”

$$q = g(r)(\dot{\theta} + \cos \theta \dot{\varphi}) \quad (16)$$

and the angular momentum vector

$$\vec{J} = \vec{x} \times \vec{p} + q \frac{\vec{x}}{r}, \quad \vec{p} = f(r)\dot{\vec{x}}. \quad (17)$$

Notice that the form of \vec{J} results from the linear representation (8) combined with the induced representation (8) [17, 8].

The remarkable result of Iwai and Katayama is that the extended Taub-NUT space (10) still admits a conserved vector, quadratic in 4-velocities, analogous to the Runge-Lenz vector of the following form

$$\vec{K} = \vec{p} \times \vec{J} + \kappa \frac{\vec{x}}{r}. \quad (18)$$

The constant κ involved in the Runge-Lenz vector (18) is

$$\kappa = -a E + \frac{1}{2} c q^2$$

where the conserved energy E is

$$E = \frac{\vec{p}^2}{2f(r)} + \frac{q^2}{2g(r)}. \quad (19)$$

The components $K_i = k_i^{\mu\nu} p_\mu p_\nu$ of the vector \vec{K} (18) involve three S-K tensors $k_i^{\mu\nu}$, $i = 1, 2, 3$ satisfying (1).

In other respects, the Poisson brackets between the components of \vec{J} and \vec{K} are similar to the relations known for the original Taub-NUT metric [1]. In particular

$$\{J_i, K_j\} = \epsilon_{ijk} K_k. \quad (20)$$

2.2 The role of the K-Y tensors

The gravitational quantum anomaly that does not exist in Ricci-flat manifolds can be also absent in manifold which do not have this property if the S-K tensors have a special structure. We refer to the situation in which the S-K tensor $k_{\mu\nu}$ can be written as a product of K-Y tensors [9].

A K-Y tensor of valence 2 is an antisymmetric tensor $f_{\mu\nu}$ satisfying the Killing equation

$$f_{\mu(\nu;\lambda)} = 0. \quad (21)$$

The integrability condition for any solution of (21) is

$$R_{\mu\nu[\sigma}{}^\tau f_{\rho]\tau} + R_{\sigma\rho[\mu}{}^\tau f_{\nu]\tau} = 0. \quad (22)$$

Now contracting this integrability condition on the Riemann tensor for any solution of (21) we get

$$f^\rho_{(\mu} R_{\nu)\rho} = 0. \quad (23)$$

Let us suppose that there exist a *square* of the S-K tensor $k_{\mu\nu}$ of the form of a K-Y tensor $f_{\mu\nu}$ [9]:

$$k_{\mu\nu} = f_{\mu\rho} f_\nu{}^\rho. \quad (24)$$

In case this should happen, the S-K equation (1) is automatically satisfied and the integrability condition (23) becomes

$$k^\rho_{[\mu} R_{\nu]\rho} = 0. \quad (25)$$

It is interesting to observe that in this last equation an antisymmetrization rather than symmetrization is involved this time as compared to (23). But this relation implies the vanishing of the commutator (6) which means that the scalar quantum anomaly does not exist for the S-K tensors which admit a decomposition in terms of K-Y tensors.

In what follows we shall exemplify the role of the Killing-Yano tensors with regards to anomalies on the Euclidean Taub-NUT space and its generalizations. The (standard) Euclidean Taub-NUT space is a hyper-Kähler manifold possessing a triplet of covariantly constant K-Y tensors, f^i , $i = 1, 2, 3$. In addition, there exist a fourth K-Y tensor, f^Y , which is not covariantly constant. The presence of this last K-Y tensor is connected with the existence of the hidden symmetries of the Taub-NUT geometry which are encapsulated in three non-trivial S-K tensors and interpreted as the components of the so-called Runge-Lenz vector of geodesic motions in this space. All these S-K tensors are products of f^Y with f^i and, moreover, the manifold is Ricci-flat since the metric tensor can be also expressed as a product of covariantly constant K-Y tensors through $f^i{}^\mu{}_{(\alpha} f^j{}_{\beta)\mu} = -2\delta_{ij} g_{\alpha\beta}$ [8]. Obviously, for this metric there are no gravitational anomalies for scalar fields.

Concerning the generalized Taub-NUT metrics, as it was done by Iwai and Katayama, it was proved that the extensions of the Taub-NUT metric do not admit K-Y tensors, even if they possess S-K tensors [19, 20]. The only exception is the original Taub-NUT metric which possesses four K-Y tensors of valence two.

Using the S-K tensor components of the Runge-Lenz vector (18) we can proceed to the evaluation of the quantum gravitational anomaly for the extended Taub-NUT metric. A direct evaluation shows that the commutator (6) does not vanish. The full explicit form of this commutator is given in the Appendix.

3 Dirac operators on generalized Taub-NUT spaces

Other sources of anomalies could be the operators of the Dirac theory on the extended Taub-NUT spaces. These have to be studied as in the case of the genuine Taub-NUT space [8] using the Cartesian coordinates instead of the spherical ones. For this reason we consider the Cartesian charts (x^1, x^2, x^3, x^4) with the line elements $d\hat{s}_K^2 = \frac{1}{b}ds_K^2$ which can be put in the form

$$d\hat{s}_K^2 = U(r)d\vec{x} \cdot d\vec{x} + V(r)(dx^4 + A_i dx^i)^2$$

where we denoted

$$U(r) = \frac{f(r)}{b}, \quad V(r) = \frac{g(r)}{b\mu^2}, \quad (26)$$

while A_i are the potentials of the Dirac magnetic monopole,

$$A_1 = -\frac{\mu}{r} \frac{x^2}{r+x^3}, \quad A_2 = \frac{\mu}{r} \frac{x^1}{r+x^3}, \quad A_3 = 0,$$

giving the magnetic field with central symmetry

$$\vec{B} = \text{rot } \vec{A} = \mu \frac{\vec{x}}{r^3}.$$

The line element of the Cartesian charts of the original Taub-NUT space have to be recovered imposing the constraints (14) and taking $\mu = \frac{a}{b} = 4m$ that assures the condition $U(r)V(r) = 1$.

In these charts it is convenient to introduce the main orbital operators of the relativistic quantum mechanics in coordinate representation. We define the momentum operators

$$P_i = -i \left(\partial_i - \sqrt{UV} A_i \partial_4 \right), \quad P_4 = -i \partial_4.$$

that give the Laplacian operator

$$\mathcal{H} = D_\mu D^\mu = -\frac{1}{U} \vec{P}^2 - \frac{1}{V} P_4^2,$$

and allow one to write the angular momentum operator as

$$\vec{L} = \vec{x} \times \vec{P} - \mu \frac{\vec{x}}{r} P_4. \quad (27)$$

The Dirac field must be defined in local frames given by tetrad fields $e(x)$ and $\hat{e}(x)$. Their components, which give us the 1-forms $\hat{e}^{\hat{\alpha}} = \hat{e}^{\hat{\alpha}}_\mu dx^\mu$ and the local derivatives $\hat{\partial}_{\hat{\nu}} = e^\mu_{\hat{\nu}} \partial_\mu$, have the usual orthonormalization properties, $g_{\alpha\beta} e^\alpha_{\hat{\mu}} e^\beta_{\hat{\nu}} = \delta_{\hat{\mu}\hat{\nu}}$, $g^{\alpha\beta} \hat{e}^{\hat{\mu}}_\alpha \hat{e}^{\hat{\nu}}_\beta = \delta^{\hat{\mu}\hat{\nu}}$, $\hat{e}^{\hat{\mu}}_\alpha e^\alpha_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}$. For the Greek indices with hats ranging from 1 to 4 the lower and upper positions are equivalent since the flat metric $\eta = 1_{4 \times 4}$

is Euclidean. Obviously the components of the metric tensor can be written as $g_{\mu\nu} = \delta_{\hat{\alpha}\hat{\beta}} \hat{e}_{\mu}^{\hat{\alpha}} \hat{e}_{\nu}^{\hat{\beta}}$. In the case of the extended Taub-NUT spaces it is convenient to choose the tetrad fields with the following non-vanishing components [21]

$$\begin{aligned}\hat{e}_j^i &= \sqrt{U} \delta_{ij}, & \hat{e}_i^4 &= \sqrt{V} A_i, & \hat{e}_4^4 &= \sqrt{V}, \\ e_j^i &= \frac{1}{\sqrt{U}} \delta_{ij}, & e_i^4 &= -\sqrt{V} A_i, & e_4^4 &= \frac{1}{\sqrt{V}}.\end{aligned}$$

The commutation relations of the derivatives $\hat{\partial}_{\hat{\nu}}$ define the Cartan coefficients $C_{\hat{\mu}\hat{\nu}}^{\hat{\sigma}} = e_{\hat{\mu}}^{\alpha} e_{\hat{\nu}}^{\beta} (\hat{e}_{\alpha,\beta}^{\hat{\sigma}} - \hat{e}_{\beta,\alpha}^{\hat{\sigma}})$, which will help us to write the spin connection in the local frames.

The next step is to choose the Dirac matrices,

$$\gamma^i = -i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix},$$

where $\mathbf{1}_2$ is the 2×2 unit matrix and σ_i are the Pauli matrices. These gamma-matrices are hermitian, satisfy $\{\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}\} = 2\delta_{\hat{\alpha}\hat{\beta}}$ and give the generators of the spinor representation [22] as

$$S^{\hat{\alpha}\hat{\beta}} = \frac{i}{4} [\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}].$$

With these ingredients we can construct the spin connection matrices,

$$\Gamma_{\sigma}^{spin} = \frac{i}{4} \hat{e}_{\sigma}^{\hat{\mu}} S^{\hat{\nu}\hat{\lambda}} (C_{\hat{\mu}\hat{\nu}\hat{\lambda}} + C_{\hat{\lambda}\hat{\mu}\hat{\nu}} + C_{\hat{\lambda}\hat{\nu}\hat{\mu}}), \quad (28)$$

involved in the structure of the covariant derivatives that give the Dirac operator

$$\mathcal{D} = \gamma^{\mu}(x) D_{\mu} = \frac{i}{\sqrt{U}} \vec{\gamma} \cdot \vec{P} + \frac{i}{\sqrt{V}} \gamma^4 P_4 + \frac{i}{2} \frac{V}{\sqrt{U}} \gamma^4 \vec{\Sigma}^* \cdot \vec{B}_{ef} \quad (29)$$

where $\gamma^{\mu}(x) = \gamma^{\hat{\alpha}} e_{\hat{\alpha}}^{\mu}(x)$,

$$\Sigma_i^* = S_i + \frac{i}{2} \gamma^4 \gamma^i, \quad S_i = -\frac{1}{2} \varepsilon_{ijk} S^{jk},$$

and $\vec{B}_{ef} = \text{rot}(\sqrt{UV} \vec{A})$. By definition the Hamiltonian operator is $H_D = \gamma^5 \mathcal{D}$. Other important observables are the generators of the global symmetry, P_4 and the whole angular momentum operator $\vec{J} = \vec{L} + \vec{S}$. One can verify that its components, \mathcal{J}_i , as well as P_4 are conserved in the sense that they commute with \mathcal{D} and H_D . This means that, as was expected, the Dirac equation is covariant under the transformations of the universal covering group of G_{iso} .

Using the Atiyah-Patodi-Singer index theorem for manifold with boundaries it was concluded that the Taub-NUT metric makes no contribution to the axial anomaly [23, 24, 25, 26]. We specify that in the Taub-NUT spaces (where $U = V^{-1}$ and $\vec{B}_{ef} = \vec{B}$) the Dirac equation $H_D \psi = E \psi$ can be analytically

solved obtaining discrete or continuous energy spectra that have no zero modes [27]. Since this space is Ricci-flat the Dirac theory has the properties (II) and (III). The four K-Y tensors of the Taub-NUT geometry give rise to four Dirac operators according to the general rule that associates to any K-Y tensor f the operator [9]

$$\mathcal{D}_f = \gamma^\mu f_\mu^\nu D_\nu - \frac{1}{6} f_{\mu\nu;\sigma} \gamma^\mu \gamma^\nu \gamma^\sigma \quad (30)$$

anticommuting with \mathcal{D} and called Dirac-type operator. The first three Dirac-type operators, \mathcal{D}_{f^i} , corresponding to the covariantly constant K-Y tensors and \mathcal{D} form a $\mathcal{N} = 4$ superalgebra and are related among themselves through continuous transformations. The fourth Dirac-type operator \mathcal{D}_{f^Y} is involved in the structure of the Runge-Lenz operator of the Dirac theory [8]. In any event, neither the standard Dirac operator nor the four Dirac-type operators produce axial anomalies.

In the case of the extended Taub-NUT spaces the problem is more complicated since the spectrum of the Dirac operator (29) can not be analytically derived and, therefore, the axial anomaly must be studied using more refined methods.

4 Index formulas on compact manifolds with boundary

Atiyah, Patodi and Singer [28] discovered an index formula for first-order differential operators on manifolds with boundary with a non-local boundary condition. Their index formula contains two terms, none of which is necessarily an integer, namely a bulk term (the integral of a density in the interior of the manifold) and a boundary term defined in terms of the spectrum of the boundary Dirac operator. Endless trouble is caused in this theory by the requirement that the metric and the operator be of "product type" near the boundary.

For Dirac operators on manifolds of the form $[l_1, l_2] \times M$, where M is closed, one can give another formula in terms of the spectral flow of the family of Dirac operators over the slices $\{t\} \times M, l_1 \leq t \leq l_2$. A related formula appears in [29] for periodic families. The rest of this section explains this index formula in the case where the metric is not of product type near the boundary.

4.1 The spectral flow

Let (M, g) be a closed Riemannian spin manifold of odd dimension with a fixed spin structure, Σ the spinor bundle and D the (self-adjoint) Dirac operator on M . Then D has discrete real spectrum accumulating towards $\pm\infty$. Moreover, the eta function

$$\eta(D, s) := \dim(\ker D) + \sum_{0 \neq \lambda \in \text{Spec } D} |\lambda|^{-s} \text{sign}(\lambda)$$

is holomorphic for $\Re(s) > \dim(M) - 1$ and extends meromorphically to \mathbb{C} . The point $s = 0$ is regular [29], and the value $\eta(D, 0)$ is by definition $\eta(D)$, the eta

invariant of D . Let g_t , $l_1 \leq t \leq l_2$, be a smooth family of Riemannian metrics on M , and D_t the Dirac operator on M with respect to g_t and the fixed spin structure. Then

$$[l_1, l_2] \ni t \mapsto f(t) := \eta(D_t)/2 \in \mathbb{R}$$

is smooth modulo \mathbb{Z} , so $t \mapsto \exp(2\pi i f(t)) \in S^1$ is smooth. By the homotopy lifting property, there exists a smooth lift \tilde{f} of $\exp(2\pi i f)$ to \mathbb{R} , the universal cover of S^1 , uniquely determined by the condition $\tilde{f}(l_1) = f(l_1)$.

$$\begin{array}{ccc} & & \mathbb{R} \\ & \nearrow \tilde{f} & \downarrow \exp \\ [l_1, l_2] & \xrightarrow{f} & S^1 \end{array}$$

From the definition, it is evident that $\tilde{f}(t) - f(t) \in \mathbb{Z}$.

Definition 1 *The spectral flow of the family $\{D_t\}_{l_1 \leq t \leq l_2}$ is*

$$\text{sf}(D_{l_1}, D_{l_2}) := f(l_2) - \tilde{f}(l_2).$$

This coincides with the original definition of the spectral flow for a path of self-adjoint Fredholm operators from [29, Section 7], which heuristically counts the net number of eigenvalues crossing 0 in the positive direction. The spectral flow is clearly a path-homotopy invariant. Now the set of Riemannian metrics is convex inside the linear space of 2-tensors. Therefore the spectral flow of the pair (D_{l_1}, D_{l_2}) is well-defined using *any* 1-parameter deformation of g_{l_1} into g_{l_2} and the associated path of Dirac operators.

4.2 A generalized APS index formula

Let

$$\Pi^\pm : \mathcal{C}^\infty(M, \Sigma) \rightarrow \mathcal{C}^\infty(M, \Sigma)$$

be the spectral projections associated to D and the intervals $[0, \infty)$, respectively $(-\infty, 0]$. More precisely, if ϕ_T is an eigenspinor of D of eigenvalue T , then

$$\Pi^+(\phi_T) = \begin{cases} \phi_T & \text{if } T \geq 0; \\ 0 & \text{otherwise;} \end{cases} \quad \Pi^-(\phi_T) = \begin{cases} \phi_T & \text{if } T \leq 0; \\ 0 & \text{otherwise.} \end{cases}$$

If X is a compact spin manifold with boundary of even dimension, then the spinor bundles $\Sigma(\partial X)$ and $\Sigma^\pm(X)|_{\partial X}$ over ∂X are canonically identified by the Clifford action of the unit normal vector field. We will need the following generalization of the Atiyah-Patodi-Singer index formula:

Theorem 2 *Let (X, g^X) be a compact spin Riemannian manifold with boundary, and*

$$\mathcal{C}^\infty(X, \Sigma^+, \Pi^-) := \{\phi \in \mathcal{C}^\infty(X, \Sigma^+); \Pi^-(\phi|_{\partial X}) = 0\}.$$

Then the operator

$$D^+ : \mathcal{C}^\infty(X, \Sigma^+, \Pi^-) \rightarrow \mathcal{C}^\infty(X, \Sigma^-)$$

is Fredholm, and

$$\text{index}(D^+) = \int_X \hat{A}(g^X) + \int_{\partial X} T\hat{A} + \frac{1}{2}\eta(D_{\partial X})$$

where $T\hat{A}$, the transgression form of \hat{A} , depends on the 2-jets of g^X at ∂X .

Proof: The fact that D^+ is Fredholm is standard in the theory of elliptic boundary value problems, see e.g., [30]. If the metric g^X were of product type near ∂X , then the Atiyah-Patodi-Singer formula [28] on X_t would read

$$\text{index}(D^+) = \int_{X_t} \hat{A}(g^X) + \frac{1}{2}\eta(D_{\partial X}) \quad (31)$$

(we use the opposite orientation for ∂X as compared to [28]). In general we cannot expect such a product structure. In a collar neighborhood defined by normal geodesic flow from ∂X , g^X takes the form

$$g^X = dt^2 + g_t$$

for $0 \leq t < \epsilon$ (see [31]), where g_t is a smooth family of metrics on ∂X . So we first deform smoothly the metric g^X into a product metric near ∂X , keeping constant the metric at the boundary and outside the fixed collar neighborhood, using a smooth function ψ :

$$h_s = dt^2 + g_{\psi(s,t)}, \quad \psi(s,t) = \begin{cases} t & \text{if } s = 0 \text{ or } t > \frac{3\epsilon}{4}; \\ 0 & \text{if } t = 0; \\ 0 & \text{if } s = 1 \text{ and } t \leq \frac{\epsilon}{2}. \end{cases}$$

The index can be computed from the action of D^+ on Sobolev spaces:

$$D^+ : H^1(X, \Sigma^+, \Pi^-) \rightarrow L^2(X, \Sigma^-).$$

The spinor bundles for different metrics are canonically identified [31]. Since by construction the vector field $\partial/\partial t$ is normal to ∂X and of length 1 for all the metrics h_s , it follows that the projection Π^- , and hence also the space $H^1(X, \Sigma^+, \Pi^-)$, do not vary with s . Let D_s^+ be the Dirac operator corresponding to the metric h_s . Then the family of bounded operators

$$D_s^+ : H^1(X, \Sigma^+, \Pi^-) \rightarrow L^2(X, \Sigma^-)$$

is norm-continuous, thus the index stays constant during the deformation. Therefore we may compute $\text{index}(D^+)$ using eq.(31) for the metric h_1 .

Next we relate the \hat{A} forms using the transgression form. Consider the connection

$$\tilde{\nabla} := ds \wedge \frac{\partial}{\partial s} + \nabla^s$$

on the bundle TX over $[0, 1] \times X$, where ∇^s is the Levi-Civita connection of the metric h_s . The curvature of $\tilde{\nabla}$ decomposes in

$$\tilde{R} = R^s + ds \wedge \theta(s)$$

where $\theta(s)$ is defined by the above equality. Therefore

$$\hat{A}(\tilde{\nabla}) = \hat{A}(\nabla^s) + ds \wedge \Theta(s)$$

and by inspection, $\Theta(s)$ depends on the 2-jets of the metric $g_{\psi(s,t)}$. Since $\hat{A}(\tilde{\nabla})$ is closed (like all characteristic forms), it follows that

$$\frac{\partial \hat{A}(\nabla^s)}{\partial s} = d\Theta(s). \quad (32)$$

Define

$$T\hat{A} := \int_0^1 \Theta(s) ds.$$

By integrating (32) on $[0, 1]$, we get $\hat{A}(h_1) - \hat{A}(h_2) = dT\hat{A}$. By Stokes's formula,

$$\int_X \hat{A}(h_1) - \int_X \hat{A}(g^X) = \int_{\partial X} T\hat{A}.$$

■

As defined, $T\hat{A}$ depends on the function ψ . For us the important conclusion is the next corollary.

Corollary 3 *Let $\{g_l^X\}_{l \in \mathbb{R}}$ be a smooth family of metrics on X , D_l^+ the associated family of Dirac operators on X , and $D_{\partial X}^l$ the induced Dirac operator on ∂X . Then there exists a smooth function $u(l)$ such that*

$$\text{index}(D_l^+) = u(l) + \frac{1}{2}\eta(D_{\partial X}^l).$$

Moreover, for $l_1 < l_2$,

$$\text{index}(D_{l_2}^+) - \text{index}(D_{l_1}^+) = \text{sf}(D_{\partial X}^{l_1}, D_{\partial X}^{l_2}).$$

Proof: Clearly $\hat{A}(g_l^X)$ depends smoothly on l . From the construction, the transgression form is also clearly smooth in l once we fix the auxiliary function ψ . We define

$$u(l) := \int_X \hat{A}(g_l^X) + \int_{\partial X} T\hat{A}(g_l^X)$$

which by Theorem 2 proves the first statement.

Using the notation from Definition 1,

$$\begin{aligned} \text{index}(D_{l_2}^+) - \text{index}(D_{l_1}^+) &= u(l_2) - u(l_1) + f(l_2) - f(l_1) \\ &= u(l_2) - u(l_1) + \tilde{f}(l_2) - \tilde{f}(l_1) \\ &\quad + \text{sf}(D_{\partial X}^{l_1}, D_{\partial X}^{l_2}). \end{aligned}$$

Thus the smooth function $u(l_2) - u(l_1) + \tilde{f}(l_2) - \tilde{f}(l_1)$ is integral-valued, and so it vanishes identically since it does at $l = l_1$. The conclusion follows by setting $l = l_2$. ■

4.3 Index theory on a cylinder

Let now g^X be a Riemannian metric on the cylinder $X := [l_1, l_2] \times M$. Endow X with the product orientation, so that $\{l_1\} \times M$ is positively oriented and $\{l_2\} \times M$ is negatively oriented inside X . Let D^+ be the chiral Dirac operator on X . For each $t \in [l_1, l_2]$ let g_t be the metric on M obtained by restricting g^X to $\{t\} \times M$. We denote by Σ_t the spinor bundle over (M, g_t) and by D_t, Π_t^\pm the Dirac operator and the spectral projections with respect to the metric g_t .

As we mentioned above, there exist canonical identifications of the spinor bundle Σ_t with $\Sigma^\pm(X)|_{\{t\} \times M}$. Consequently it makes sense to denote by ϕ_t the restriction of a positive spinor from X to $\{t\} \times M$.

Theorem 4 *Let $X = [l_1, l_2] \times M$ be a product spin manifold with a smooth metric g^X as above. Set*

$$\mathcal{C}^\infty(X, \Sigma^+, \Pi^-) := \{\phi \in \mathcal{C}^\infty(X, \Sigma^+); \Pi_{l_1}^+ \phi_{l_1} = 0, \Pi_{l_2}^- \phi_{l_2} = 0\}.$$

Then

$$\text{index}(D^+ : \mathcal{C}^\infty(X, \Sigma^+, \Pi^-) \rightarrow \mathcal{C}^\infty(X, \Sigma^-)) = \text{sf}(D_{l_1}, D_{l_2}).$$

Note that the projection $\Pi_{l_2}^-$ equals $\Pi_{l_2}^+$ for the opposite orientation on $\{l_2\} \times M$, which is the one induced from X .

Proof: Deform the metric g^X in a neighborhood of $\{l_1\} \times M$ to a product metric as in the proof of Theorem 2. As explained there, this deformation does not change the index. The spectral flow is also unchanged (we noted that it depends only on the two metrics on the ends). For $l_1 < t \leq l_2$ let $X_t := [l_1, t] \times M \subset X$. Then Corollary 3 gives

$$\begin{aligned} \text{index}(D_t^+) &= u(t) + f(t) - f(l_1) \\ &= u(t) + \tilde{f}(t) - \tilde{f}(l_1) + \text{sf}(l_1, t). \quad \text{by Def. 1} \end{aligned}$$

Note that both the \hat{A} volume form and the transgression $T\hat{A}$, hence also $u(t)$, vanish for t near l_1 in the product region. Thus the smooth function $u(t) + \tilde{f}(t) - \tilde{f}(l_1)$ takes values in \mathbb{Z} , on the other hand both $u(t)$ and $\tilde{f}(t) - \tilde{f}(l_1)$ vanish at $t = l_1$, so $u(t) + \tilde{f}(t) - \tilde{f}(l_1)$ vanishes identically. The conclusion follows by setting $t = l_2$. ■

Note that a similar statement concerning spectral boundary value problems appears in [32].

5 Harmonic spinors over Berger spheres

Since the cohomology groups of S^3 vanish in dimensions 1 and 2, there exists a unique spin structure on S^3 . Let D_λ be the Dirac operator corresponding to the Berger metric g_λ defined in eq. (8). Recall that D_λ is essentially self-adjoint (in L^2) with discrete spectrum.

Lemma 5 *For $\lambda < 2$, D_λ does not admit harmonic spinors.*

Proof: It is easy to compute the scalar curvature of g_λ . This is done for instance in [25]. Namely, $\kappa(g_\lambda)$ is constant on S^3 , $\kappa(g_\lambda) = (4 - \lambda^2)/12$. In particular $\kappa(g_\lambda)$ is positive for $\lambda < 2$. Lichnerowicz's formula proves then that $\ker D_\lambda = 0$. ■

More generally, Hitchin [18] computed the eigenvalues of D_λ . In this paper we are only interested in eigenvalues close to 0. Let us recall Hitchin's result in this case.

Theorem 6 ([18]) *Let*

$$\Lambda(\lambda) := \{(p, q) \in \mathbb{N}^{*2}; \lambda^2 = 2\sqrt{(p-q)^2 + 4\lambda^2 pq}\}.$$

Then

$$\dim \ker(D_\lambda) = N(\lambda) := \sum_{(p,q) \in \Lambda(\lambda)} p + q.$$

If $N(\lambda) > 0$ there exists $\epsilon > 0$ such that for $|t - \lambda| < \epsilon$, the "small" eigenvalues of D_t are given by families

$$T(t, p, q) := \frac{t}{2} - \sqrt{\frac{(p-q)^2}{t^2} + 4pq}, \quad (p, q) \in \Lambda(\lambda) \quad (33)$$

with multiplicity $p + q$.

In particular, harmonic spinors appear first for $\lambda = 4$ where the kernel of D_4 is two-dimensional. Moreover, the set of those $\lambda \in (0, \infty)$ for which $N(\lambda) \neq 0$ is discrete. For $l > 0$ set

$$S(l) := \sum_{\lambda \leq l} N(\lambda). \quad (34)$$

Of course the sum is finite for finite l .

Corollary 7 *The spectral flow of the family $\{D_t\}_{t \in [l_1, l_2]}$ of Berger Dirac operators equals $S(l_2) - S(l_1)$.*

Proof: By differentiating eq.(33) we see that the function $t \rightarrow T(t, p, q)$ is strictly increasing, so the spectral flow of the family $\{D_t\}$ across $t = \lambda$ is precisely $N(\lambda)$. ■

6 The extended Taub-NUT metric

Let us consider the extended Taub-NUT metric ds_K^2 on $\mathbb{R}^4 \setminus \{0\} \simeq (0, \infty) \times S^3$ given by eq. (12) in terms of the Berger metrics. We clearly need $a + br > 0$ for all $r > 0$ so we ask that $a \geq 0, b > 0$. Also $d > 0$ seems reasonable in order for the metric to be defined for large r , and even $c > -2\sqrt{d}$ so that $1 + cr + dr^2 > 0$ for all $r > 0$. However there seems to be no reason to ask $c \geq 0$, so $\lambda(r)$ may become large for certain values of r .

In mathematical terms, axial anomalies translate to Dirac operators with non-vanishing index. We are interested in the chiral Dirac operator on an annular piece of $\mathbb{R}^4 \setminus \{0\}$. First set $X_{l_1, l_2} := [l_1, l_2] \times S^3 \subset \mathbb{R}^4 \setminus \{0\}$ with the induced extended Taub-NUT metric.

Theorem 8 *The index of D^+ over (X_{l_1, l_2}, ds_K^2) with the APS boundary condition is*

$$\text{index}(D^+) = S(\lambda(l_2)) - S(\lambda(l_1))$$

where the function S is given by (34).

Proof: By Theorem 4 the index is equal to the spectral flow of the pair of boundary Dirac operators. Now the metrics on the boundary spheres are constant multiples of the Berger metrics $g_{\lambda(l_1)}$, respectively $g_{\lambda(l_2)}$. The spectral flow of a path of conformal metrics (even with non-constant conformal factor) vanishes by the conformal invariance of the space of harmonic spinors [18]. Thus the spectral flow can be computed using the pair of metrics $g_{\lambda(l_1)}$ and $g_{\lambda(l_2)}$. The conclusion follows from Corollary 7. ■

It is a number-theoretic question to determine $S(\lambda)$ in general. We can give however some conditions which entail the vanishing of the index.

Corollary 9 *If $c > -\frac{\sqrt{15d}}{2}$ then the extended Taub-NUT metric does not contribute to the axial anomaly on any annular domain (i.e., the index of the Dirac operator with APS boundary condition vanishes).*

Proof: The hypothesis implies that $\lambda(r) < 4$ for all $r > 0$. From the remark following Theorem 6 we see that $S(\lambda(l_1)) = S(\lambda(l_2)) = 0$. ■

We obtain as a particular case the vanishing of the index from [25]. Another case when the index vanishes is when l_1 and l_2 are either small or large enough so that both $\lambda(l_1)$ and $\lambda(l_2)$ are less than 4.

The singularity at the origin of the extended Taub-NUT metric is removable, in the sense that there exists a smooth extension to \mathbb{R}^4 .

Theorem 10 *For $l > 0$ let X_l be the ball $X_l := \{r \leq l\} \subset \mathbb{R}^4$, endowed with the generalized Taub-NUT metric ds_K^2 . Then*

$$\text{index}(D^+) = S(\lambda(l)).$$

Proof: Deform the metric on X_l smoothly into the standard metric ds^2 on the ball X_l . Now $ds^2 = dr^2 + r^2 d\sigma^2$ is a warped product near $r = l$, so we can further deform the warping factor to be constant near $r = l$. Let h_0 be the resulting metric and D_0^+ its Dirac operator. The restriction of h_0 to ∂X_l is a multiple of $g_1/4$, the standard metric on S^3 . By Corollaries 3 and 7,

$$\text{index}(D^+) = \text{index}(D_0^+) + S(\lambda(l))$$

since $S(1) = 0$. We use the APS index formula (31) to compute $\text{index}(D_0^+) = 0$. Indeed, the eta invariant of the standard sphere vanishes since the spectrum is symmetric around 0, while the \hat{A} volume form of a warped product metric vanishes by the conformal invariance of the Pontrjagin forms. ■

7 Unbounded domains

There appear two other possibilities to construct index problems for the metric ds_K^2 . First we have the mixed APS- L^2 boundary condition on $[l, \infty) \times S^3$; and secondly we have the L^2 index problem on R^4 .

The metric ds_K^2 is of *fibred cusp* type at infinity in the sense of [33]. Indeed, with the change of variables $x = 1/r$ near $r = \infty$, we have

$$ds_K^2 = (ax + b) \left(\frac{dx^2}{x^4} + \frac{g_H}{x^2} + \frac{1}{d + cx + x^2} g_V \right). \quad (35)$$

It is impossible to present here Φ -operators and the associated Φ -calculus $\Psi_\Phi(\mathbb{R}^4)$; we refer the interested reader to [33, 34, 35, 36]. The index of Dirac operators for *exact* Φ metrics was computed in [34] under a tameness assumption on the kernel of the family of vertical Dirac operators. Unfortunately, (35) is not exact in the sense of [34] because of the factor $d + cx + x^2$. A general but less precise index formula for fully elliptic Φ -operators was given in [35] and then improved in [36], where the case of a fibration over S^1 is studied in detail.

A priori it is not at all clear if D^+ is Fredholm in $L^2(\mathbb{R}^4)$, although from Theorem 10, the limit as $l \rightarrow \infty$ of the index on X_l exists and equals 0. A general principle of Melrose's analysis of pseudodifferential algebras asserts that an operator in such an algebra is Fredholm on appropriate Sobolev spaces if and only if it is *fully elliptic*. Before explaining what this is, note that the corresponding statement for Φ -operators is proved in [33].

7.1 Fully elliptic Φ -operators

Let X denote the radial compactification of \mathbb{R}^4 . There exists first a notion of principal symbol for Φ -operators, living on the Φ -cotangent bundle, a smooth extension of $T\mathbb{R}^4$ to X .

There exists additionally a "boundary symbol" map called the normal operator, which is a star-morphism

$$\mathcal{N} : \Psi_\Phi(\mathbb{R}^4) \rightarrow \Psi_{\text{sus}(\Phi T^* S^2) - \phi}(S^3)$$

with values in the *suspended algebra* [37], an algebra of parameter-dependent operators along the fibers of the Hopf fibration. A Φ -operator is called fully elliptic if both its principal symbol and its normal operator are invertible.

Theorem 11 *The Dirac operator on (\mathbb{R}^4, ds_K^2) is not fully elliptic.*

Proof: The principal symbol of D^2 is precisely the metric ds_K^2 , which extends to a Riemannian metric on $\Phi T^* X$. This shows that D is elliptic.

Let $u : \mathbb{R}^4 \rightarrow (0, \infty)$ be a function which near $x = 0$ equals $ax + b$. Define a metric h on \mathbb{R}^4 conformal to ds_K^2 by $h := \frac{ds_K^2}{u}$. Then the Dirac operators of the metrics h and ds_K^2 are related by [18, Prop. 1.3]:

$$D_h = u^{5/4} D u^{-3/4}.$$

Now notice that $u(x) = \sqrt{d} > 0$ for $x = 0$, and recall that the map \mathcal{N} is multiplicative. Thus we see that the normal operators of D_h and D are simultaneously invertible. We focus in the rest of the proof on D_h .

We want to show that D_h is not fully elliptic, so we only look at the region $x \leq l$. Let $v(x) := \sqrt{d + cx + x^2}$, so that

$$h = \frac{dx^2}{x^4} + \frac{g_H}{x^2} + \frac{1}{v^2(x)}g_V.$$

Let I, J, K be the three vector fields on S^3 , viewed as the quaternion unit sphere, corresponding to the three distinguished complex structures. Let $V_j, 0 \leq j \leq 3$, be the following Φ -vector fields on X near the boundary:

$$V_0 := x^2 \partial_x, \quad V_1 := v(x)I, \quad V_2 := xJ, \quad V_3 := xK.$$

These vector fields form an orthonormal frame p in ${}^\Phi T X$, parallel in the direction of V_0 (with respect to the Levi-Civita covariant derivative of the metric h). We use this frame (more precisely, one of its lifts \tilde{p} to the Spin bundle) to trivialize the spinor bundle. After some computations, we get

$$\mathcal{N}(D_h)(0) = c^1(V_1 + \sqrt{d}c^2c^3)$$

in the above trivialization. Each integral curve \mathcal{C} of V_1 has length $2\pi/\sqrt{d}$; let t be the arc-length parameter on \mathcal{C} . Let ψ be a spinor with

$$(V_1 + \sqrt{d}c^2c^3)\psi = 0. \tag{36}$$

We can assume that ψ is a section of Σ^+ , the other case is similar. The restriction of ψ to \mathcal{C} is given by a curve

$$[0, 2\pi/\sqrt{d}) \ni t \mapsto \psi(t) \in \mathbb{C}^2,$$

where the two factors of \mathbb{C} are the $\pm i$ eigenspaces of c^2c^3 . In other words, $\psi(t) = (\psi_+(t), \psi_-(t))$ with $c^2c^3\psi_\pm(t) = \pm i\psi_\pm(t)$. Then eq. (36) reduces to

$$\psi'(t) \pm i\sqrt{d}\psi_\pm(t) = 0$$

and this equation does have solutions, namely $\psi_\pm(t) = e^{\mp it\sqrt{d}}\psi(0)$. The point is that the solution is periodic of period equal to the length of \mathcal{C} . Equivalently, ψ_\pm can be any smooth section in the complex line bundle over S^2 associated to the Hopf principal S^1 -bundle $S^3 \rightarrow S^2$ and the ± 1 representations of S^1 on \mathbb{C} . \blacksquare

Thus our Dirac operator is not Fredholm on $L^2(\mathbb{R}^4, \Sigma)$. However, it may still have a finite-dimensional kernel and cokernel. We leave open the question of determining the index in this case, but we conjecture it to be 0.

The same argument shows that the Dirac operator on $[l, \infty) \times S^3$ with mixed APS- L^2 boundary conditions is not Fredholm either. Again, we leave open the question of determining its index.

8 Concluding remarks

There is a relationship between the absence of gravitational anomalies and the existence of K-Y tensors.

For scalar fields, the decomposition (24) of S-K tensors in terms of K-Y tensors guarantees the absence of gravitational anomalies. Otherwise operators constructed from symmetric tensors are in general a source of anomalies proportional to the Ricci tensors.

However for the axial anomaly the role of K-Y tensors is not so obvious. The topological aspects are more important and the existence of K-Y tensors is not directly related to anomalies.

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Appendix A

Explicit evaluation of the gravitational anomaly for generalized Taub-NUT metrics

In order to evaluate the commutator (6) involving the components of the S-K tensors corresponding to the Runge-Lenz vector (18), we limit ourselves to give only the components of the third S-K $k_3^{\mu\nu}$ tensor in spherical coordinates. Its non vanishing components are:

$$\begin{aligned}
k_3^{rr} &= -\frac{ar \cos \theta}{2(a+br)} \\
k_3^{r\theta} &= k_3^{\theta r} = \frac{\sin \theta}{2} \\
k_3^{\theta\theta} &= \frac{(a+2br) \cos \theta}{2r(a+br)} \\
k_3^{\varphi\varphi} &= \frac{(a+2br) \cot \theta \csc \theta}{2r(a+br)} \\
k_3^{\varphi\chi} &= k_3^{\chi\varphi} = -\frac{(2a+3br+br \cos(2\theta) \csc^2 \theta)}{4r(a+br)} \\
k_3^{\chi\chi} &= \frac{(a-adr^2+br(2+cr)+(a+2br)) \cot^2 \theta \cos \theta}{2r(a+br)}.
\end{aligned}$$

Again, just to exemplify, we write down from the commutator (6) only the

function which multiplies the covariant derivative D_r :

$$\frac{3r \cos \theta}{4(a + br)^3(1 + cr + dr^2)^2} \cdot \\ ((-a^2(c^2 + 2cdr + 2d(-1 + dr^2)) - 2abr(c^2 + 2cdr + 2d(-1 + dr^2)) + \\ b^2(2 + c^2r^2 + 6dr^2 + 2cr(2 + dr^2))) .$$

Of course, as it is expected the commutator (6) vanishes for the standard Euclidean Taub-NUT metric, i.e. the constant a, b, c, d are constrained by (14).

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